# MOTION OF A VORTEX NEAR THE BOUNDARY BETWEEN TWO HEAVY FLUIDS

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The motion of a vortex beneath the surface of a heavy fluid has been discussed in both linear [1, 2] and nonlinear [3-5] formulation. The density of the upper medium is neglected, which makes it possible to replace the continuity of pressure during transition through the boundary between the media by constancy of the pressure at the boundary of the heavy fluid. In this paper, the problem is solved in a general nonlinear formulation, including the mutual effects of media motion, and the vortex can be in either the upper or lower medium. Steady-state motion of a vortex of given intensity near the boundary between two heavy fluids is discussed in terms of a model of an ideal and incompressible medium. Approximate expressions are obtained for the boundary.

In the plane of the complex variable $z = x + iy$ we consider steady-state motion of a medium consisting of two ideal incompressible fluids with densities $\rho_+$ and $\rho_-$ in a gravitational field with the potential gy.

Let the medium at infinity move along the x axis at a velocity $v_\infty$ and let there be a vortex of intensity $\Gamma$ at the point $z = ih$ (Fig. 1).

We designate by $D_+$ and $D_-$ the flow regions of the fluids with respective densities $\rho_+$ and $\rho_-$; the complex velocities of the fluids in these regions are $u_+$ and $u_-$. We assume that the fluid with lower density ($\rho_+ < \rho_-$) is above the boundary L, the equation for which in parametric form is

$$z=f(\zeta)\ (\zeta=e^{i\varphi},\ 0 \leqslant \varphi \leqslant 2\pi).$$

Under the assumptions made, the problem reduces to a determination of the function $\bar{u}_-(z)$, which is analytic everywhere in the region $D_-$, of the function $\bar{u}_+(z)$, which is analytic in $D_+$ with the exception of the point $z = ih$ where it has a pole of first order, and of the shape of the boundary $z = f(\zeta)$ under the following conditions:

$$\operatorname{Im}\,[i\bar{u}_\pm\,(f(\zeta))\ f'\,(\zeta)\ \zeta]=0; \tag{1}$$

$$\operatorname{Im}(z)-\frac{\mathrm{Fr}}{2}=-\frac{\mathrm{Fr}}{2}\,[|u_-(z)|^2+\varepsilon\,(|u_-(z)|^2-|u_+(z)|^2)],\quad z=f(\zeta); \tag{2}$$

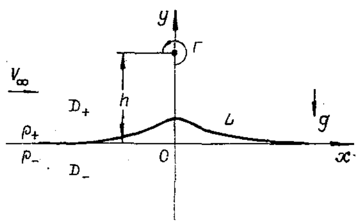


Fig. 1

$$\lim_{|z|\to\infty} u_\pm(z)=1, \tag{3}$$

where

$$\mathrm{Fr}=v_\infty^2/gb;\quad \varepsilon=\rho_+/(\rho_- - \rho_+).$$

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The conditions (1)-(3), respectively, represent the condition of kinematic consistency of the flows, the condition of continuity of pressure during the transition through the boundary of the media, and the condition of damping of perturbed velocities at infinity.

We reduce this nonlinear problem containing the three unknown functions $\bar{u}_+(z)$, $\bar{u}_-(z)$, and $f(\zeta)$ to a problem involving a single unknown function.

Let the function $f(w)$ conformally map the interior of the circle $|w| < 1$ onto the region $D_+$ so that the point $w = i$ transforms into the point of infinity and the point $w = 0$ to the point $z = ih$. The function $q(\psi)$ conformally maps the interior of the circle $|\psi| < 1$ on the region $D_-$ where the point $\psi = i$ transforms into the point of infinity and the point $\psi = 0$ onto the point $z = -ih$. We introduce the function $q_1(w)$, which is analytic inside the circle $|w| < 1$, such that

$$|q_1(\zeta)|=1, \quad |\zeta|=1, \; f(\zeta)=q(q_1(\zeta)), \; |\zeta|=1.$$

We consider the functions

$$G_+(w)=\bar{u}_+(f(w))f'(w); \; G_-(w)=\bar{u}_-(q(q(w)))f'(w). \tag{4}$$

In accordance with the singularities of the functions $\bar{u}_+(z)$, $\bar{u}_-(z)$, and $f(w)$ the function $G_+(w)$ must have a pole of first order at the point $w = 0$ and a pole of second order at the point $w = i$, and the function $G_-(w)$ must be analytic everywhere within the circle $|w| < 1$ and have a pole of second order at the point $w = i$. The condition (1) for these functions will be

$$\mathrm{Re}[G_\pm(\zeta)\cdot\zeta]=0, \quad |\zeta|=1. \tag{5}$$

Taking the singularities of the functions $G_+(w)$ and $G_-(w)$ into account together with the conditions (3) and (5), we write them in the following way:

$$G_+(w)=\frac{a_2}{(w-i)^2}-\frac{ia_1}{w}; \quad G_-(w)=\frac{a_2}{(w-i)^2}, \tag{6}$$

where $a_1$ and $a_2$ are arbitrary real numbers. The pressure continuity condition (2) makes it possible to obtain an equation for determining the mapping function:

$$\left(\mathrm{Im}(f)-\frac{\mathrm{Fr}}{2}\right)|f'|^2=-\frac{\mathrm{Fr}}{2}\left[|G_-|^2+\varepsilon\left(|G_-|^2-|G_+|^2\right)\right]. \tag{7}$$

Thus the original problem is reduced to a search for a solution to Eq. (7) for the function $f(w)$ and to a determination of the constants $a_1$ and $a_2$.

We shall look for a function $f(w)$ in the form

$$f(w)=\sum_{k=0}^{\infty} f_k(w)\,\varepsilon^k. \tag{8}$$

Then, equating coefficients of like powers of $\varepsilon$, we obtain the following recursion relations for the determination of the functions $f_k(w) = (k = 0, 1, \ldots)$:

$$\left(\mathrm{Im}(f_0)-\frac{\mathrm{Fr}}{2}\right)|f_0'|^2=-\frac{\mathrm{Fr}}{2}|G_-|^2; \tag{9}$$

$$|f_0'|^2\,\mathrm{Im}(f_1)+g_1\,\mathrm{Im}(f_0)=-\frac{\mathrm{Fr}}{2}\left(|G_-|^2-|G_+|^2\right); \tag{10}$$

$$|f_0'|^2\,\mathrm{Im}(f_k)=-\sum_{p=1}^{k} g_p\,\mathrm{Im}(f_{k-p}), \quad k=2,3,\ldots,$$

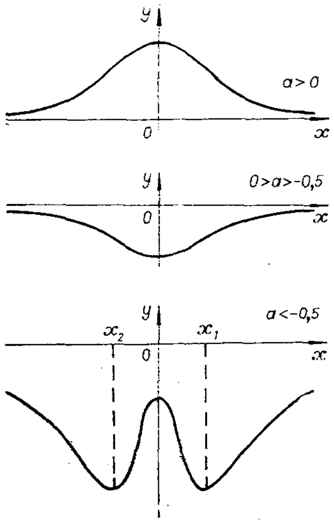


Fig. 2

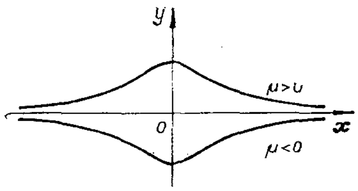


Fig. 3

where

$$g_p = \sum_{s=0}^{p} f_s' \bar{f}_{p-s}', \quad p = 0, 1, \ldots .$$

Without discussing in detail the question of the convergence of the series (8), we point out that the parameter $\varepsilon$ is very small ($\varepsilon \sim 10^{-3}$) for such media as air–water. We therefore confine ourselves to the first two terms, $f_0$ and $f_1$, in the expansion (8) assuming that the modulus of the functions $f_k$ does not increase rapidly as k increases.

Using condition (3) and the fact that the point w = i transforms into the point at infinity and the point w = 0 into the point z = ih, we obtain a solution of Eqs. (9) and (10) in the form

$$f(w) = -\frac{c}{w-i} - ih + 2i\nu\left(\frac{3}{4}a + 1\right) - \nu(a+1)w - i\frac{\nu a}{4}w^2, \tag{11}$$

where

$$\nu = \varepsilon \,\mathrm{Fr}\, a_1/h; \quad a = a_1/h; \quad c = -2h + 2\nu\left(\frac{3}{4}a + 1\right). \tag{12}$$

Knowing the mapping function $f(w)$, we find the velocity of the fluid in the region $D_+$ from Eqs. (4) and (6),

$$\bar{u}_+(z) = 1 + \frac{a_1}{i(z-ih)} - \frac{a_1}{i(z+ih)} + \nu\left[-\frac{h\left(\frac{3}{2}a^2 + 3a + 2\right)}{(z+ih)^2} - \frac{3ih^2 a(a+2)}{(z+ih)^3} + \frac{2h^3 a^2}{(z+ih)^4}\right].$$

The first three terms determine the fluid velocity for motion of a vortex of intensity $\Gamma = 2\pi a_1$ in the neighborhood of a screen, and the remaining terms take into account the change in the shape of the boundary.

Using similar arguments, we show that the function $q(\psi)$ has the form

$$q(\psi) = -\frac{d}{\psi - i} + ih + 2i\nu\left(\frac{3}{4}a + 1\right) - \nu(a+1)\psi - i\frac{\nu a}{4}\psi^2, \tag{13}$$

where

$$d = 2h + 2\nu\left(\frac{3}{4}a + 1\right),$$

and the velocity of the fluid in the region $D_-$ is

$$\bar{u}_-(z) = 1 + \nu\left[\frac{(2+a)h}{(z-ih)^2} - \frac{2iah^2}{(z-ih)^3}\right].$$

Using Eq. (11) or (13), the equation for the boundary between the fluids can be written in the form

$$y = \frac{2\nu h^2}{x^2 + h^2}\left(1 + \frac{ah^2}{x^2 + h^2}\right),$$

where $a$ and $\nu$ are given by Eqs. (12), where $a_1 = \Gamma/2\pi$.

The variation in the shape of the line L as a function of the parameter $a$, which characterizes the intensity of the vortex and its closeness to the boundary, is shown in Fig. 2. When $a > 0$, the vortex increases the velocity of the upper fluid in the neighborhood of the boundary, leading to a decrease in the pressure on the lower fluid and a rise of that fluid. When $-0.5 < a < 0$, the vortex decelerates the motion of the upper fluid near the boundary with a consequent depression of the fluid immediately beneath the vortex. The marked change in the shape of the line L when $a < -0.5$, namely, the appearance of two minima $x_{1,2} = \pm h\sqrt{-2a-1}$ at which the value of the function is unchanged as the parameter $a$ decreases $[y(x_{1,2}) = -\mathrm{Fr}\cdot\varepsilon/2]$, is caused by the creation of two critical points on the boundary between the fluids:

$$\zeta_{1,2} = i\left(1 - \frac{a_2}{2a_1}\right) \pm \sqrt{\frac{a_2}{2a_1}\left(2 - \frac{a_2}{2a_1}\right)}.$$

Now let the vortex be in the denser fluid (at the point $z = -ih$). Then the functions $G_+(w)$ and $G_-(w)$ change places,

$$G_+(w) = \frac{a_2}{(w-i)^2}; \quad G_-(w) = \frac{a_2}{(w-i)^2} - \frac{ia_1}{w},$$

however, Eq. (9) will not have so simple a solution as in the previous case. If one represents the mapping function in the form of a series in powers of $a(|a| \ll 1)$

$$f(w) = \sum_{k=0}^{\infty} f_k(w)\, a^k,$$

we obtain equations for the determination of the functions $f_0(w)$ and $f_1(w)$ similar to Eqs. (9) and (10), which when solved yield

$$f(w) = -\frac{c}{w-i} + ih + i\mu - \frac{\mu}{2}w.$$

Here

$$\mu = 2\mathrm{Fr}(1+\varepsilon)a_1/h, \quad c = 2h + \mu.$$

In this case, the fluid velocity in the region $D_-$ has the form

$$\bar{u}_-(z) = 1 + \frac{a_1}{i(z+ih)} - \frac{a_1}{i(z-ih)} - \frac{\mu(a_1+h)}{(z-ih)^2} + \frac{2\mu iha_1}{(z-ih)^3},$$

the mapping function $q(\psi)$ and the fluid velocity in the region $D_+$ are, respectively,

$$q(\psi) = \frac{2h-\mu}{\psi - i} - ih + i\mu - \frac{\mu}{2}\psi;$$

$$\bar{u}_+(z) = 1 - \frac{\mu h}{(z+ih)^2}.$$

The line of density discontinuity of the fluids has the shape of a single wave, the curvature of which depends on the sign of the circulation (Fig. 3)

$$y = \frac{\mu h^2}{x^2 + h^2}.$$

## LITERATURE CITED


1. M. V. Keldysh, "Notes on certain motions of a heavy fluid," Tekh. Zametki Tsentr. Aéro-Gidrodinam. Inst., No. 52 (1935).
2. N. E. Kochin, "Wave resistance of bodies submerged in a fluid," in: Collected Works [in Russian], Vol. 2, Izd. Akad. Nauk SSSR, Moscow (1949).
3. A. I. Nekrasov, "Point vortex beneath the surface of a heavy ideal fluid in plane-parallel flow," in: Collected Works, [in Russian], Vol. 2, Izd. Akad. Nauk SSSR, Moscow (1962).
4. A. M. Ter-Krikorov, "Exact solution of the problem of motion of a vortex beneath the surface of a fluid," Izv. Akad. Nauk SSSR, Ser. Mat., 22, No. 2 (1958).
5. I. G. Filippov, "Motion of a vortex beneath the surface of a fluid," Prikl. Mat. Mekh., 25, No. 2 (1961).